On the periodic fundamental solutions of the Stokes equations and their application to viscous flow past a cubic array of spheres

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(Received 15 June 1958)

Spatially periodic fundamental solutions of the Stokes equations of motion for a viscous fluid past a periodic array of obstacles are obtained by use of Fourier series. It is made clear that the divergence of the lattice sums pointed out by Burgers may be rescued by taking into account the presence of the mean pressure gradient. As an application of these solutions the force acting on any one of the small spheres forming a periodic array is considered. Cases for three special types of cubic lattice are investigated in detail. It is found that the ratios of the values of this force to that given by the Stokes formula for an isolated sphere are larger than 1 and do not differ so much among these three types provided that the volume concentration of the spheres is the same and small. The method is also applied to the two-dimensional flow past a square array of circular cylinders, and the drag on one of the cylinders is found to agree with that calculated by the use of elliptic functions.

1. Introduction

The study of the flow of a viscous fluid past a periodic array of spheres is very important from the theoretical and practical viewpoints. As far as the author is aware, however, many treatments of this problem have been made only by considering some artificial models, even in the case of the slow motion. This may be partly due to the fact that standard methods of constructing the solutions of the Stokes equations, by summing up the induced velocity of the isolated particles, encounter the difficulty of divergence pointed out already by Burgers (1941). Burgers proposed the so-called diffuse field of force to overcome this difficulty. This idea is extended by Brinkman (1947, 1948) and independently by Debye & Bueche (1948), who considered a model consisting of a sphere separated by an infinitely thin shell from a porous medium. Their main concern was in the study of the sedimentation of suspensions and the flow through porous media. The statistical treatment due to Kynch (1954, 1956) using the shielded potential belongs to this category.

Making use of a method analogous to the approximate method due to Wigner & Seitz (1933), Uchida (1949) investigated the flow past a simple cubic lattice of spheres. Although the satisfaction of the periodic condition is not perfect, it may be regarded as one of the direct attacks on our problem. There is also Kawaguchi's ingenious treatment (1958) of replacing the field by the flow past a sphere in a frictionless circular pipe.

Recently, Tamada & Fujikawa (1957) investigated the two-dimensional flow through an infinite row of parallel circular cylinders on the basis of Oseen's equations and showed that the drag on one of the cylinders tends to the Stokes type in the limit of the small Reynolds numbers. Inspired by their results the author (1958*a*) discussed the flow through a thin screen and obtained an exact solution of the Stokes equation for a periodic series of flat plates set perpendicularly to the uniform flow. Kuwabara (1958) and Miyagi (1958) also treated respectively the flow past a row of parallel flat plates and the flow past a row of circular cylinders on the basis of the Stokes equations. Making use of elliptic functions the author (1958*b*) also determined the two-dimensional flow past a doubly periodic array of circular cylinders.

Taking into account these successful results it is natural for us to suspect that solutions for the case of an array of spheres exist and can be obtained by some kind of direct attack.

In this paper periodic fundamental solutions of the Stokes equations of motion for a viscous incompressible fluid past a periodic array of obstacles are given by use of Fourier series. It is made clear that the divergence pointed out by Burgers may be avoided by taking into account the presence of the mean pressure gradient which is present in our problem. As a simple example, the flow past a cubic lattice of small spheres is examined for three types of lattice (simple, face-centred, body-centred) and the force acting on any one of the spheres is determined as a function of volume concentration. It is found that the ratios of the values of this force to that given by the Stokes formula for an isolated sphere are larger than 1 and do not differ much for these three types so long as the volume concentration of spheres is the same and small. The method is also applied to the two-dimensional flow past a square array of circular cylinders, and the drag on any one of the cylinders is found to agree with that calculated by the author (1958*b*), using elliptic functions.

2. Basic equations and periodic fundamental solutions

Let us consider the steady motion of an incompressible viscous fluid past a periodic array of small obstacles with their centres at

$$\mathbf{r}_n = n_1\mathbf{a}^{(1)} + n_2\mathbf{a}^{(2)} + n_3\mathbf{a}^{(3)} \quad (n_1, n_2, n_3 = 0, \pm 1, \pm 2, \ldots), \tag{2.1}$$

where $\mathbf{a}^{(1)}$, $\mathbf{a}^{(2)}$ and $\mathbf{a}^{(3)}$ are the basic vectors determining the unit cell of the array.

According to the procedure of Lamb (1932) and Burgers (1938) the fundamental solutions of our problem are obtained by solving the following Stokes equation of motion and continuity equation

$$\mu\Delta\mathbf{V} = \operatorname{grad} p + \mathbf{F}\sum_n \delta(\mathbf{r}-\mathbf{r}_n) \quad \left(\Delta = \frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right), \tag{2.2}$$

$$\operatorname{div}\mathbf{V} = 0, \tag{2.3}$$

where $\mathbf{V}$ is the velocity, μ the viscosity, p the pressure, $\mathbf{F}$ the force acting on one of the obstacles, (x_1, x_2, x_3) are the Cartesian co-ordinates of the position vector $\mathbf{r}$, and $\delta(\mathbf{r}-\mathbf{r_n})$ denotes Dirac's delta function defined by the conditions

$$\int_\tau \delta(\mathbf{r}-\mathbf{r_n})\,d\mathbf{r} = \begin{cases} 1 & \text{for} \quad \tau \in \mathbf{r_n} \\ 0 & \text{for} \quad \tau \notin \mathbf{r_n}, \end{cases} \tag{2.4}$$

and

$$\delta(\mathbf{r}-\mathbf{r_n}) = 0 \quad \text{for} \quad \mathbf{r} \neq \mathbf{r_n}.$$

Taking into account the periodicity of the flow field we expand $\mathbf{V}$ and $-\operatorname{grad} p$ in Fourier series:

$$\mathbf{V} = \sum_{\mathbf{k}} \mathbf{V_k}\, e^{-2\pi i(\mathbf{k}.\mathbf{r})}, \tag{2.5}$$

$$-\operatorname{grad} p = \sum_{\mathbf{k}} \mathbf{P_k}\, e^{-2\pi i(\mathbf{k}.\mathbf{r})}, \tag{2.6}$$

where

$$\mathbf{k} = n_1 \mathbf{b}^{(1)} + n_2 \mathbf{b}^{(2)} + n_3 \mathbf{b}^{(3)} \tag{2.7}$$

are vectors in the reciprocal lattice, which satisfy

$$\mathbf{k}.\mathbf{a}^{(j)} = n_j \quad (j = 1, 2, 3). \tag{2.8}$$

Making use of (2.7) and (2.8) we find the basic vectors $\mathbf{b}^{(1)}$, $\mathbf{b}^{(2)}$ and $\mathbf{b}^{(3)}$ in the reciprocal lattice to be

$$\mathbf{b}^{(1)} = \frac{[\mathbf{a}^{(2)} \times \mathbf{a}^{(3)}]}{\tau_0}, \quad \mathbf{b}^{(2)} = \frac{[\mathbf{a}^{(3)} \times \mathbf{a}^{(1)}]}{\tau_0}, \quad \mathbf{b}^{(3)} = \frac{[\mathbf{a}^{(1)} \times \mathbf{a}^{(2)}]}{\tau_0}, \tag{2.9}$$

where

$$\tau_0 = \mathbf{a}^{(1)}.[\mathbf{a}^{(2)} \times \mathbf{a}^{(3)}] \tag{2.10}$$

stands for the volume of the unit cell in the physical space.

Multiplying (2.2) and (2.3) by $e^{2\pi i(\mathbf{k}.\mathbf{r})}/\tau_0$ and integrating over a unit cell in physical space, we obtain

$$-4\pi^2 \mu k^2 \mathbf{V_k} = -\mathbf{P_k} + \frac{\mathbf{F}}{\tau_0} \quad (k^2 = \mathbf{k}.\mathbf{k}), \tag{2.11}$$

$$(\mathbf{k}.\mathbf{V_k}) = 0. \tag{2.12}$$

$\mathbf{P_k}$ satisfies the relation

$$\mathbf{P_k} \times \mathbf{k} = 0, \tag{2.13}$$

which can be proved by taking the curl of (2.6).

We begin by considering the terms for which $\mathbf{k} = 0$. Equation (2.11) yields

$$\mathbf{P}_0 = \frac{\mathbf{F}}{\tau_0}, \tag{2.14}$$

which means that the force acting on an obstacle is balanced by the mean pressure gradient of the fluid. Disregard of this relation would induce paradoxical results e.g. the divergence of $\mathbf{V}_0$ as already shown by Burgers.

Taking the scalar product of (2.11) with $\mathbf{k}$ we obtain, for $\mathbf{k} \neq 0$,

$$(\mathbf{k}.\mathbf{P_k}) = \frac{1}{\tau_0}(\mathbf{k}.\mathbf{F}) = (\mathbf{k}.\mathbf{P}_0) \tag{2.15}$$

or, making use of (2.12) and (2.13),

$$\mathbf{P_k} = \frac{(\mathbf{k}.\mathbf{F})\,\mathbf{k}}{\tau_0 k^2} \quad (\mathbf{k} \neq 0). \tag{2.16}$$

Substitution of (2.16) in (2.11) gives

$$\mathbf{V}_{\mathbf{k}} = \frac{1}{4\pi^2\mu\tau_0}\left[\frac{(\mathbf{k}.\mathbf{F})\,\mathbf{k}}{k^4} - \frac{\mathbf{F}}{k^2}\right] \quad (\mathbf{k} \neq 0). \tag{2.17}$$

Equations (2.5) and (2.6) with (2.14) to (2.17) are the periodic fundamental solutions of the Stokes equations for the flow past a periodic array of obstacles. Their components in Cartesian co-ordinates are given by

$$v_j = v_{0j} - \frac{1}{4\pi\mu}\left(F_j S_1 - \sum_{l=1}^{3} F_l \frac{\partial^2 S_2}{\partial x_l \partial x_j}\right), \tag{2.18}$$

$$-(\operatorname{grad} p)_j = \frac{F_j}{\tau_0} - \frac{1}{4\pi}\sum_{l=1}^{3} F_l \frac{\partial^2 S_1}{\partial x_l \partial x_j}, \tag{2.19}$$

where S_1 and S_2 are given by

$$S_1 = \frac{1}{\pi\tau_0}\sum_{\mathbf{k}\neq 0}{}' \frac{e^{-2\pi i(\mathbf{k}.\mathbf{r})}}{k^2}, \quad S_2 = -\frac{1}{4\pi^3\tau_0}\sum_{\mathbf{k}\neq 0}{}' \frac{e^{-2\pi i(\mathbf{k}.\mathbf{r})}}{k^4}, \tag{2.20}$$

and may be proved to be the solutions of the following equations:

$$\Delta S_2 = S_1 \tag{2.21}$$

and

$$\Delta S_1 = -4\pi\left[\sum_{\mathbf{n}} \delta(\mathbf{r} - \mathbf{r}_{\mathbf{n}}) - \frac{1}{\tau_0}\right], \tag{2.22}$$

by use of finite Fourier transforms.

It is important that S_1 is not harmonic even in the domain excluding $\mathbf{r}_{\mathbf{n}}$, because of the presence of the term $4\pi/\tau_0$. This makes clear the reason why the standard method of obtaining S_1 by summing up fundamental solutions of the Laplace equation for isolated particles has encountered difficulties. S_1 is equivalent to the electrostatic potential of a lattice composed of positive unit charges surrounded by the cloud of uniform negative charge which neutralizes them. It is interesting to note that the presence of this uniform charge corresponds to the presence of the mean pressure gradient in our problem (see (2.2)) and rescues the lattice sums from divergence.

3. Evald's technique for evaluating S_1 and S_2

In order to satisfy the boundary condition on the surface of an obstacle, it is necessary to evaluate S_1 and S_2 at small values of $\mathbf{r}$. A technique for this purpose was presented by Evald (1921) in the calculation of the Madelung energy of the ionic crystals in terms of S_1, and is summarized by Born & Misra (1940) in a convenient form.

We start with an integral representation for $1/k^{2m}$:

$$\frac{1}{k^{2m}} = \frac{\pi^m}{\Gamma(m)}\int_0^\infty e^{-\pi k^2\beta}\,\beta^{m-1}\,d\beta. \tag{3.1}$$

Multiplying by $e^{-2\pi i(\mathbf{k}.\mathbf{r})}$ and summing with respect to $\mathbf{k}$, except for $\mathbf{k} = 0$, we have

$$\begin{aligned}\sigma_m = \sum_{\mathbf{k}\neq 0}{}' \frac{e^{-2\pi i(\mathbf{k}.\mathbf{r})}}{k^{2m}} &= \frac{\pi^m}{\Gamma(m)}\sum_{\mathbf{k}\neq 0}{}' \int_0^\infty e^{-\pi k^2\beta - 2\pi i(\mathbf{k}.\mathbf{r})}\,\beta^{m-1}\,d\beta \\ &= \frac{\pi^m}{\Gamma(m)}\int_0^\infty \beta^{m-1}\left[\sum_{\mathbf{k}} e^{-\pi k^2\beta - 2\pi i(\mathbf{k}.\mathbf{r})} - 1\right]d\beta. \end{aligned}\tag{3.2}$$

Let us split the integral into two parts, one to be taken from 0 to α, and the other from α to ∞, and then apply Evald's theta transformation formula

$$\sum_{\mathbf{k}} e^{-\pi k^2\beta-2\pi i(\mathbf{k}.\mathbf{r})} = \frac{\tau_0}{\beta^{\frac{3}{2}-\lambda}} \sum_{\mathbf{n}} e^{-\pi(\mathbf{r}-\mathbf{r}_\mathbf{n})^2/\beta} \tag{3.3}$$

to the integral from 0 to α, where α is a moderate constant and

$$\lambda = \begin{cases} 0 & \text{for the three-dimensional case,} \\ \frac{1}{2} & \text{for the two-dimensional case.} \end{cases} \tag{3.4}$$

We get

$$\sigma_m = \frac{\pi^m \alpha^m}{\Gamma(m)} \left[\tau_0 \alpha^{-\frac{3}{2}+\lambda} \sum_{\mathbf{n}} \phi_{-m+\frac{1}{2}-\lambda}\left(\frac{\pi(\mathbf{r}-\mathbf{r}_\mathbf{n})^2}{\alpha}\right) - \frac{1}{m} + \sum_{\mathbf{k}\neq 0}{}' e^{-2\pi i(\mathbf{k}.\mathbf{r})} \phi_{m-1}(\pi\alpha \mathbf{k}^2)\right], \tag{3.5}$$

where we have put $\beta = \alpha/\xi$ in the first integral and $\beta = \alpha\xi$ in the second, and $\phi_\nu(x)$ is the incomplete Γ-function

$$\phi_\nu(x) = \int_1^\infty \xi^\nu e^{-x\xi} d\xi. \tag{3.6}$$

The function $\phi_\nu(x)$ satisfies the recurrence formulae

$$\phi'_\nu = -\phi_{\nu+1}, \quad x\phi_\nu = e^{-x} + \nu\phi_{\nu-1}, \tag{3.7}$$

and is tabulated in Born & Misra's paper. In particular

$$\phi_0(x) = \frac{e^{-x}}{x}, \quad \phi_{-\frac{1}{2}}(x) = \frac{\sqrt{\pi}}{\sqrt{x}} \operatorname{erfc}(\sqrt{x}), \quad \phi_{-1}(x) = -E_i(-x), \tag{3.8}$$

where

$$\operatorname{erfc}(\xi) = \frac{2}{\sqrt{\pi}} \int_\xi^\infty e^{-\xi^2} d\xi \tag{3.9}$$

is the complementary error function and

$$-E_i(-\xi) = \int_\xi^\infty \frac{e^{-\xi}}{\xi} d\xi \tag{3.10}$$

denotes the exponential integral.

These functions tend to zero rapidly as $x \to \infty$, and

$$\phi_{-\frac{1}{2}}(x) = \frac{\sqrt{\pi}}{\sqrt{x}} - 2 + \frac{2x}{3} + O(x^2), \quad \phi_{-\frac{3}{2}}(x) = 2 - 2\sqrt{(\pi x)} + 2x + O(x^2), \tag{3.11}$$

$$\phi_{-1}(x) = -\gamma - \log x + x + O(x^2), \quad \phi_{-2}(x) = 1 - x(1-\gamma-\log x) + O(x^2), \tag{3.12}$$

as $x \to 0$, where $\gamma = 0{\cdot}577215\ldots$ is Euler's constant.

Making use of these expressions we can evaluate the values of S_1 and S_2, etc., which are given in the following sections.

4. The case of a lattice of small spheres

As an application of the fundamental solutions (2.18) and (2.19) we consider the case of periodic array of spheres of equal radius a which is very small compared with the mutual distances of the spheres.

In order to satisfy the remaining boundary conditions on a sphere

$$\mathbf{V} = 0 \quad \text{at} \quad r = \sqrt{(|\mathbf{r}|^2)} = a, \tag{4.1}$$

we have only to know the behaviour of the fundamental solutions as $r \to 0$. Making use of (3.5) and (3.11) we obtain for S_1, S_2 and $\partial^2 S_2/\partial x_j \partial x_l$:

$$S_1 = \frac{\sigma_1}{\pi\tau_0} = \frac{1}{r} - c + O(r^2), \tag{4.2}$$

$$c = \frac{2}{\sqrt{\alpha}} + \frac{\alpha}{\tau_0} - \frac{1}{\sqrt{\alpha}} \sum_{\mathbf{n}\neq 0}' \phi_{-\frac{1}{2}}\left(\frac{\pi \mathbf{r}_\mathbf{n}^2}{\alpha}\right) - \frac{\alpha}{\tau_0} \sum_{\mathbf{k}\neq 0}' \phi_0(\pi\alpha\mathbf{k}^2), \tag{4.2'}$$

$$S_2 = -\frac{\sigma_2}{4\pi^3\tau_0} = \frac{r}{2} - c_2 + O(r^2), \tag{4.3}$$

$$c_2 = \frac{1}{4\pi}\left[2\sqrt{\alpha} - \frac{\alpha^2}{2\tau_0} + \sqrt{\alpha} \sum_{\mathbf{n}\neq 0}' \phi_{-\frac{3}{2}}\left(\frac{\pi \mathbf{r}_\mathbf{n}^2}{\alpha}\right) + \frac{\alpha^2}{\tau_0} \sum_{\mathbf{k}\neq 0}' \phi_1(\pi\alpha\mathbf{k}^2)\right] \tag{4.3'}$$

$$\frac{\partial^2 S_2}{\partial x_j \partial x_l} = -\frac{x_j x_l}{2r^3} + \delta_{jl}\left(\frac{1}{2r} - c_{jj}\right) + O(r^2) \quad \left(\delta_{jl} = \begin{cases} 1 & (j = l) \\ 0 & (j \neq l) \end{cases}\right), \tag{4.4}$$

$$c_{jj} = \frac{1}{\sqrt{\alpha}} - \frac{1}{2\sqrt{\alpha}} \sum_{\mathbf{n}\neq 0}' \left[\phi_{-\frac{1}{2}}\left(\frac{\pi \mathbf{r}_\mathbf{n}^2}{\alpha}\right) - \frac{2\pi x_{\mathbf{n}j}^2}{\alpha} \phi_{\frac{1}{2}}\left(\frac{\pi \mathbf{r}_\mathbf{n}^2}{\alpha}\right)\right] - \frac{\pi\alpha^2}{\tau_0} \sum_{\mathbf{k}\neq 0}' \mathbf{k}_j^2 \phi_1(\pi\alpha\mathbf{k}^2), \tag{4.4'}$$

where $x_{\mathbf{n}j}$ and $\mathbf{k}_j$ denote respectively x_j-component of $\mathbf{r}_\mathbf{n}$ and $\mathbf{k}$.

Burgers has shown that a good approximation to $\mathbf{F}$ can be obtained if we determine $\mathbf{F}$ from the condition that the mean velocity on the sphere vanishes, i.e.

$$\langle \mathbf{V} \rangle = \frac{1}{4\pi a^2} \int_{r=a} \mathbf{V}\, dS = 0. \tag{4.5}$$

Inserting (4.2) to (4.4′) into (2.18), neglecting all terms of higher order than (a^2) and using

$$\langle x_j x_l \rangle = \frac{a^2}{3} \delta_{jl}, \tag{4.6}$$

we have

$$4\pi\mu v_{0j} = \left(\frac{2}{3a} - c + c_{jj}\right) F_j + O(a^2), \tag{4.7}$$

i.e.

$$F_j = \frac{6\pi\mu a v_{0j}}{1 - \kappa_j a} + O(a^3), \tag{4.8}$$

where κ_j is a constant determined by the basic vectors $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$:

$$\kappa_j = \tfrac{3}{2}(c - c_{jj}) = \frac{3}{2}\left[\frac{1}{\sqrt{\alpha}} + \frac{\alpha}{\tau_0} - \frac{1}{2\sqrt{\alpha}} \sum_{\mathbf{n}\neq 0}' \left(\phi_{-\frac{1}{2}} + \frac{2\pi x_{\mathbf{n}j}^2}{\alpha}\phi_{\frac{1}{2}}\right) - \frac{\alpha}{\tau_0} \sum_{\mathbf{k}\neq 0}' (\phi_0 - \pi\alpha\mathbf{k}_j^2 \phi_1)\right]. \tag{4.9}$$

As a typical case we consider the cubic array of spheres. Now, there are three types of cubic lattice:

(1) Simple cubic lattice (S.C.L)

$$\left.\begin{matrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{matrix}\right\} = h \begin{Bmatrix}(1, 0, 0)\\ (0, 1, 0)\\ (0, 0, 1)\end{Bmatrix}, \quad \tau_0 = h^3, \qquad \left.\begin{matrix}\mathbf{b}_1\\ \mathbf{b}_2\\ \mathbf{b}_3\end{matrix}\right\} = 1/h \begin{Bmatrix}(1, 0, 0)\\ (0, 1, 0)\\ (0, 0, 1)\end{Bmatrix} \tag{4.10}$$

(2) Body-centred cubic lattice (B.C.L)

$$\left.\begin{matrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{matrix}\right\} = \tfrac{1}{2}h \begin{Bmatrix}(\ 1, & 1, & -1)\\ (-1, & 1, & 1)\\ (\ 1, & -1, & 1)\end{Bmatrix}, \quad \tau_0 = \tfrac{1}{2}h^3, \qquad \left.\begin{matrix}\mathbf{b}_1\\ \mathbf{b}_2\\ \mathbf{b}_3\end{matrix}\right\} = 1/h \begin{Bmatrix}(1, 1, 0)\\ (0, 1, 1)\\ (1, 0, 1)\end{Bmatrix} \tag{4.11}$$

(3) Face-centred cubic lattice (F.C.L)

$$\left.\begin{matrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{matrix}\right\} = \tfrac{1}{2}h\left\{\begin{matrix}(1,1,0)\\(0,1,1)\\(1,0,1)\end{matrix}\right\},\quad \tau_0 = \tfrac{1}{4}h^3,\qquad \left.\begin{matrix}\mathbf{b}_1\\ \mathbf{b}_2\\ \mathbf{b}_3\end{matrix}\right\} = 1/h\left\{\begin{matrix}(\;1, & 1, & -1)\\(-1, & 1, & 1)\\(1, & -1, & 1)\end{matrix}\right\} \tag{4.12}$$

According to their symmetry with respect to three co-ordinates axes it is obvious that

$$\left\langle\frac{\partial^2 S_2}{\partial x_1^2}\right\rangle = \left\langle\frac{\partial^2 S_2}{\partial x_2^2}\right\rangle = \left\langle\frac{\partial^2 S_2}{\partial x_3^2}\right\rangle = \tfrac{1}{3}\langle \Delta S_2\rangle = \tfrac{1}{3}\langle S_1\rangle \tag{4.13}$$

and

$$c_{11} = c_{22} = c_{33} = \tfrac{1}{3}c. \tag{4.14}$$

Introducing (4.14) into (4.9) we obtain

$$\kappa_1 = \kappa_2 = \kappa_3 = c = -\lim_{r\to 0}\left(S_1 - \frac{1}{r}\right). \tag{4.15}$$

The values of c have been calculated by many authors (e.g. Emersleben 1923) in connexion with the determination of the Madelung constants, and are given in table 1. In this table the values of ca are also given in terms of the volume concentration of spheres:

$$c_\tau = \frac{4\pi}{3}\frac{a^3}{\tau_0}. \tag{4.16}$$

	ch	$ca/\sqrt[3]{c_\tau}$	bh^2
S.C.L	2·8373	1·7601	0·19457
B.C.L	$3{\cdot}639_2$	$1{\cdot}791_8$	$0{\cdot}120_0$
F.C.L	$4{\cdot}584_8$	$1{\cdot}791_7$	$0{\cdot}213_1$

TABLE 1

5. Refinement of the approximation in the case of cubic arrays

In order to refine the rough approximation in the previous section we have only to determine the coefficients of the complementary functions which are derived from S_1 and S_2 by successive differentiation and which are to be added to (2.18).

As one of the most simple cases we consider the case of the cubic array. Let us assume that the mean flow is parallel to the x_1-axis without loss of generality. In this case we can take

$$v_1 = U_0 - \frac{1}{4\pi\mu}\left[\mathbf{G}\left(S_1 - \frac{\partial^2 S_2}{\partial x_1^2}\right) + \mathbf{H}\frac{\partial^2 S_1}{\partial x_1^2}\right], \tag{5.1}$$

$$v_2 = \frac{1}{4\pi\mu}\left[\mathbf{G}\frac{\partial^2 S_2}{\partial x_1 \partial x_2} - \mathbf{H}\frac{\partial^2 S_1}{\partial x_1 \partial x_2}\right], \tag{5.2}$$

$$v_3 = \frac{1}{4\pi\mu}\left[\mathbf{G}\frac{\partial^2 S_2}{\partial x_1 \partial x_3} - \mathbf{H}\frac{\partial^2 S_1}{\partial x_1 \partial x_3}\right], \tag{5.3}$$

$$-(\operatorname{grad} p)_j = \frac{F_j}{\tau_0} - \frac{1}{4\pi}\operatorname{grad}\mathbf{G}\frac{\partial S_1}{\partial x_1}, \tag{5.4}$$

making use of (2.18) and (2.19) and taking into account the symmetry, where **G** and **H** stand for differential operators

$$\mathbf{G} = \sum_{m,n,p=0}^{\infty} A_{mnp} \frac{\partial^{2m+2n+2p}}{\partial x_1^{2m} \partial x_2^{2n} \partial x_3^{2p}}, \quad \mathbf{H} = \sum_{m,n,p=0}^{\infty} B_{mnp} \frac{\partial^{2m+2n+2p}}{\partial x_1^{2m} \partial x_2^{2n} \partial x_3^{2p}}. \tag{5.5}$$

A_{mnp} and B_{mnp} are symmetric with respect to n and p, and some of them can be taken to be zero on account of the relations (2.21) and (2.22).

In particular, U_0, A_{000} and B_{000} are intimately related with the mean velocity U and the force **F** acting on a sphere as follows. The force acting on a sphere is shown to be

$$F_1 = A_{000} = A, \quad F_2 = 0, \quad F_3 = 0, \tag{5.6}$$

by comparison of (5.4) and (2.19). On the other hand, the mean velocity U in the direction of x_1-axis is given by the surface integral

$$U = \frac{1}{h^2} \int_{-\frac{1}{2}h}^{\frac{1}{2}h} \int_{-\frac{1}{2}h}^{\frac{1}{2}h} v_1 \, dx_2 \, dx_3, \tag{5.7}$$

taken over a surface which is bounded by a rectangle of area $h \times h$ perpendicular to x_1-axis and is outside every sphere. Making use of the relations

$$S_1 - \frac{\partial^2 S_2}{\partial x_1^2} = \left(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) S_2, \tag{5.8}$$

$$\frac{\partial^2 S_1}{\partial x_1^2} = \frac{4\pi}{\tau_0} - \left(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) S_1, \tag{5.9}$$

for $\mathbf{r} \neq \mathbf{r}_n$, derived from (2.21) and (2.22) and the periodicity of $\mathbf{G}[\partial S_{1,2}/\partial x_j]$ and $\mathbf{H}[\partial S_{1,2}/\partial x_j]$ $(j = 2, 3)$ with respect to x_j, we obtain

$$U = \frac{1}{h^2} \int_{-\frac{1}{2}h}^{\frac{1}{2}h} \int_{-\frac{1}{2}h}^{\frac{1}{2}h} \left(U_0 - B_{000} \frac{1}{\mu \tau_0} \right) dx_2 \, dx_3 = U_0 - \frac{B}{\mu \tau_0} \quad (B = B_{000}). \tag{5.10}$$

We note that the mean flow is not U_0, but has an additional term which is shown to be of the order of c_τ (see (5.22)). The x_2- and x_3-components of the mean velocity are easily proved to be zero.

Let us proceed to the determination of these constants from the boundary condition (4.1) on a sphere of small radius a.

For this purpose we expand $S_1 - 1/r$ and $S_2 - \frac{1}{2}r$ in spherical harmonics with their common centre at $r = 0$ as follows:

$$S_1 = \frac{1}{r} - c + \frac{2\pi}{3\tau_0} r^2 + \sum_{n=2}^{\infty} \sum_{m=0}^{m \leq \frac{1}{4}n} a_{nm} \, Y_{2n}^{4m}(x_1, x_2, x_3) \tag{5.11}$$

$$S_2 = \frac{r}{2} - c_2 - \frac{c}{6} r^2 + \frac{\pi r^4}{30\tau_0} + \sum_{n=2}^{\infty} \sum_{m=0}^{m \leq \frac{1}{4}n} (b_{nm} + \tilde{a}_{nm} r^2) \, Y_{2n}^{4m}(x_1, x_2, x_3), \tag{5.12}$$

where use is made of Hobson's theorem (1931, p. 161), together with the cubic symmetry of these functions, and

$$Y_n^m(x_1, x_2, x_3) = r^n P_n^m(\cos\theta) \cos m\phi, \tag{5.13}$$

with $$x_1 = r\cos\theta, \quad x_2 = r\sin\theta\cos\phi, \quad x_3 = r\sin\theta\sin\phi. \tag{5.14}$$

Also, c and c_2 are given by (4.2′), (4.3′) and table 1, and

$$\left.\begin{matrix} a_{nm} \\ b_{nm} \end{matrix}\right\} = \frac{\epsilon_m \cdot 2^{2n}(2n)!\,(2n-4m)!}{(4n)!\,(2n+4m)!}\left[Y_{2n}^{4m}\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3}\right)\left\{\begin{matrix} S_1-1/r \\ S_2-r/2 \end{matrix}\right\}\right]_{r=0}$$

$$(\epsilon_0 = 1, \epsilon_m = 2 \text{ for } m>0), \quad (5.15)$$

$$\tilde{a}_{nm} = \frac{1}{2(4n+3)}a_{nm}. \quad (5.16)$$

Introducing into (4.1) the expressions (5.1) to (5.3) with (5.6) and (5.10) to (5.12) and equating to zero the coefficients of $P_0, P_2, \ldots$ in v_1 and $P_2' e^{i\phi}, \ldots$ in v_2+iv_3 respectively we obtain the following simultaneous equations

$$\left[\frac{2}{3a}-\frac{2c}{3}+\frac{4\pi}{9\tau_0}\right]F-\frac{8\pi}{3\tau_0}B+\left[\frac{8\pi}{15\tau_0}-24b+O(a^2)\right]A_{100}+O(a^7) = 4\pi\mu U, \quad (5.17)$$

$$\left[\frac{1}{3a}-\left(\frac{8\pi}{45\tau_0}+12b\right)a^2+O(a^4)\right]F+\left[\frac{2}{a^3}+O(a^2)\right]B+\left[\frac{2}{7a^3}+O(a^2)\right]A_{100}+O(a^5) = 0, \quad (5.18)$$

$$\left[\frac{1}{6a}-\left(\frac{4\pi}{45\tau_0}-4b\right)a^2\right]F+\left[\frac{1}{a^3}+O(a^2)\right]B-\left[\frac{3}{7a^3}+O(a^2)\right]A_{100}+O(a^5) = 0, \quad (5.19)$$

$$\cdots \quad \cdots \quad \cdots$$

$$b \equiv b_{20}.$$

Comparing the main terms in these equations we see at once that, at most,

$$F = O(a), \quad B = O(a^3), \quad A_{mnp} = O(a^{4(m+n+p)+1}), \quad B_{mnp} = O(a^{4(m+n+p)+3}). \quad (5.20)$$

Solving (5.17) to (5.19), we obtain

$$A_{100} = \tfrac{35}{2}ba^5F+O(a^8), \quad (5.21)$$

$$B = -\frac{1}{6}\left[1-\left(\frac{8\pi}{15\tau_0}+21b\right)a^3\right]a^2F+O(a^8), \quad (5.22)$$

$$F = 6\pi\mu a U/Q, \quad (5.23)$$

where

$$Q = 1-ca+\frac{4\pi a^3}{3\tau_0}-\left(\frac{16\pi^2}{45\tau_0^2}+630b^2\right)a^6+O(a^8) \quad (5.23')$$

and

$$b = b_{20} = -\frac{\pi^3}{15}\left[\alpha^{-\frac{5}{2}}\sum_{\mathbf{n}\neq 0}{}'(x_{\mathbf{n}1}^4-3x_{\mathbf{n}1}^2x_{\mathbf{n}2}^2)\,\phi_{\frac{5}{2}}\left(\frac{\pi\mathbf{r}_\mathbf{n}^2}{\alpha}\right)+\frac{\alpha^2}{\tau_0}\sum_{\mathbf{k}\neq 0}{}'(\mathbf{k}_1^4-3\mathbf{k}_1^2\mathbf{k}_2^2)\,\phi_1(\pi\alpha\mathbf{k}^2)\right]. \quad (5.24)$$

The numerical values of b are given in table 1. In terms of c_τ equations (5.23′) are rewritten as follows:

$$Q = \left\{\begin{matrix} 1-1{\cdot}7601\sqrt[3]{c_\tau}+c_\tau-1{\cdot}5593\,c_\tau^2+\ldots & (\text{s.c.l.}) \\ 1-1{\cdot}791_8\sqrt[3]{c_\tau}+c_\tau-0{\cdot}329_2\,c_\tau^2+\ldots & (\text{b.c.l.}) \\ 1-1{\cdot}791_7\sqrt[3]{c_\tau}+c_\tau-0{\cdot}302_0\,c_\tau^2+\ldots & (\text{f.c.l.}) \end{matrix}\right\}. \quad (5.25)$$

Making use of (5.25) we can calculate the values of F for three types of cubic lattice, the results being shown in table 2 and figure 1, where $O(c_\tau^\nu)$ stands for the values calculated by retaining the terms of the order of c_τ^ν.

It is found that the values of F are larger than $6\pi\mu aU$, i.e. the value for an isolated sphere, and do not differ much for the three types of cubic lattice, provided that the values of c_τ are the same and small. This conclusion will be applicable also to other types of lattice.

	$\sqrt[3]{c_\tau}$	a/h	$Q:O(\sqrt[3]{c_\tau})$	$Q:O(c_\tau)$	$Q:O(c_\tau^2)$	$Q^{-1}:O(c_\tau^2)$
S.C.L	0	0·00000	1·0000	1·0000	1·0000	1·0000
	0·05	0·03102	0·9120	0·9121	0·9121	1·096
	0·10	0·06204	0·8240	0·8250	0·8250	1·212
	0·20	0·12407	0·6480	0·6560	0·6559	1·525
	0·30	0·18611	0·4720	0·4990	0·4978	2·009
	0·40	0·24814	0·2960	0·3600	0·3536	2·83
	0·50	0·31018	0·1199	0·2449	0·2206	4·5
	0·60	0·37221	—	0·1599	0·0872	12
B.C.L	0	0·00000	1·0000	1·0000	1·0000	1·0000
	0·05	0·02462	0·9104	0·9105	0·9105	1·098
	0·10	0·04924	0·8208	0·8218	0·8218	1·217
	0·20	0·09847	0·6416	0·6496	0·6496	1·539
	0·30	0·14771	0·4625	0·4895	0·4892	2·044
	0·40	0·19695	0·2833	0·3473	0·3459	2·89
	0·50	0·24619	0·1041	0·2291	0·2240	4·47
	0·60	0·29542	—	0·1409	0·1256	8·0
	0·70	0·34466	—	0·0888	0·0500	20
F.C.L	0	0·00000	1·0000	1·0000	1·0000	1·0000
	0·05	0·01954	0·9104	0·9105	0·9105	1·098
	0·10	0·03908	0·8208	0·8218	0·8128	1·217
	0·20	0·07816	0·6417	0·6497	0·6496	1·539
	0·30	0·11724	0·4625	0·4895	0·4893	2·044
	0·40	0·15632	0·2833	0·3473	0·3461	2·89
	0·50	0·19540	0·1042	0·2292	0·2244	4·47
	0·60	0·23448	—	0·1410	0·1269	7·9
	0·70	0·27356	—	0·0888	0·0533	19

TABLE 2

6. Two-dimensional case

The two-dimensional case, i.e. the flow past an array of circular cylinders of radius $r = \sqrt{(x_1^2+x_2^2)} = a$, can be treated in a similar manner.

Putting $\lambda = \frac{1}{2}$ in (3.5), etc., we obtain, for example,

$$S_1 = \frac{\sigma_1}{\pi\tau_0} = \sum_{\mathbf{n}} \phi_{-1}\left(\frac{\pi(\mathbf{r}-\mathbf{r}_\mathbf{n})^2}{\alpha}\right) - \frac{\alpha}{\tau_0} + \frac{\alpha}{\tau_0}\sum_{\mathbf{k}\neq 0}' e^{-2\pi i(\mathbf{k}.\mathbf{r})}\phi_0(\pi\alpha\mathbf{k}^2) \tag{6.1}$$

$$= -\log(\pi r^2/\alpha) - \bar{c} + O(r^2),$$

$$\bar{c} = \gamma + \frac{\alpha}{\tau_0} - \sum_{\mathbf{n}\neq 0}' \phi_{-1}\left(\frac{\pi\mathbf{r}_\mathbf{n}^2}{\alpha}\right) - \frac{\alpha}{\tau_0}\sum_{\mathbf{k}\neq 0}' \phi_0(\pi\alpha\mathbf{k}^2), \tag{6.1'}$$

$$S_2 = -\frac{\sigma_2}{4\pi^3\tau_0} = -\frac{\alpha}{4\pi}\left[\sum_{\mathbf{n}} \phi_{-2}\left(\frac{\pi(\mathbf{r}-\mathbf{r}_\mathbf{n})^2}{\alpha}\right) - \frac{\alpha}{2\tau_0} + \frac{\alpha}{\tau_0}\sum_{\mathbf{k}\neq 0}' e^{-2\pi i(\mathbf{k}.\mathbf{r})}\phi_1(\pi\alpha\mathbf{k}^2)\right], \tag{6.2}$$

$$S_1 - \frac{\partial^2 S_2}{\partial x_1^2} = \frac{\partial^2 S_2}{\partial x_2^2} = -\tfrac{1}{2}\log\frac{\pi r^2}{\alpha} - \frac{x_2^2}{r^2} - \frac{\gamma}{2} + \Sigma_2 + O(r^2), \tag{6.3}$$

$$\Sigma_2 = \tfrac{1}{2}\sum_{\mathbf{n}}'\left[\phi_{-1}\left(\frac{\pi\mathbf{r}_\mathbf{n}^2}{\alpha}\right) - \frac{2\pi x_{\mathbf{n}2}^2}{\alpha}\phi_0\left(\frac{\pi\mathbf{r}_\mathbf{n}^2}{\alpha}\right)\right] + \frac{\pi\alpha^2}{\tau_0}\sum_{\mathbf{k}}' \mathbf{k}_2^2\phi_1(\pi\alpha\mathbf{k}^2). \tag{6.3'}$$

From these results we can calculate the drag per unit length of a circular cylinder in the array:

$$F_1 \doteqdot \frac{4\pi\mu v_{01}}{\langle S_1 - \partial^2 S_2/\partial x_1^2\rangle} \doteqdot \frac{8\pi\mu v_{01}}{\log(\alpha/\pi a^2) - 1 - \gamma + 2\Sigma_2}. \tag{6.4}$$

In particular, for a square array of period h we get the formula

$$F_1 \doteqdot \frac{4\pi\mu U_1}{\frac{1}{2}\langle S_1\rangle}$$
$$= \frac{8\pi\mu U_1}{\log(h^2/\pi a^2) - 1 - \gamma + \sum_{\mathbf{n}\neq 0}' [\phi_0(\pi\mathbf{n}^2) + \phi_{-1}(\pi\mathbf{n}^2)]} = \frac{4\pi\mu U_1}{\log(h/a) - 1\cdot 3105\ldots}, \tag{6.5}$$

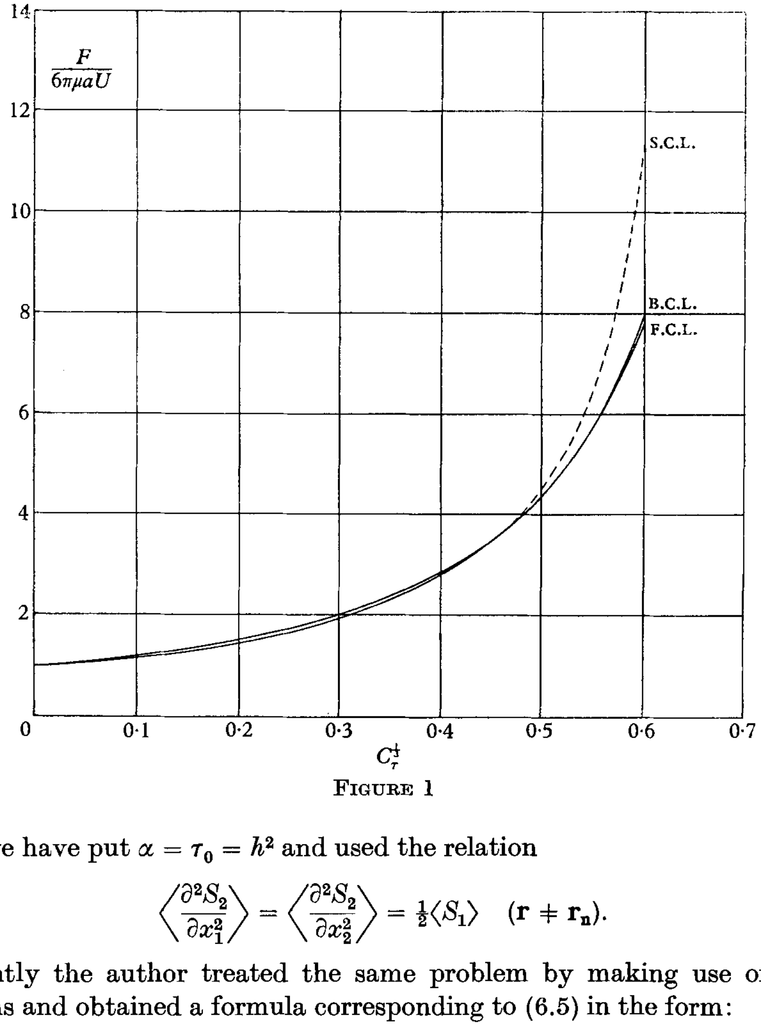

FIGURE 1

where we have put $\alpha = \tau_0 = h^2$ and used the relation

$$\left\langle \frac{\partial^2 S_2}{\partial x_1^2} \right\rangle = \left\langle \frac{\partial^2 S_2}{\partial x_2^2} \right\rangle = \tfrac{1}{2}\langle S_1\rangle \quad (\mathbf{r} \neq \mathbf{r_n}). \tag{6.6}$$

Recently the author treated the same problem by making use of elliptic functions and obtained a formula corresponding to (6.5) in the form:

$$F_1 = \frac{4\pi\mu U_1}{\log(h/a) - \tilde{c} + (\pi a^2/h^2) + O(a^4/h^4)}, \tag{6.7}$$

where $$\tilde{c} = \log\left[2\pi \prod_{n=1}^{\infty} (1 - e^{-2n\pi})^2\right] + \tfrac{1}{4} - \tfrac{1}{4}\pi + \tfrac{1}{2}\pi \sum_{n=1}^{\infty} \frac{1}{\sinh^2 n\pi} = 1\cdot 3105\ldots. \tag{6.8}$$

It will be seen that the two formulae (6.5) and (6.7) are in perfect agreement with each other to the order here considered.

This problem was suggested by Prof. Ko Tamada. The author wishes to express his cordial thanks to Prof. Tamada, Prof. Isao Imai, and Prof. Susumu Tomotika for their continual encouragement and helpful discussions throughout this work. The author's thanks are also due to the Japan Ministry of Education for a grant in aid for fundamental scientific research.

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